Problem 2.22

Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch $V(r)$.

Solution

An electrostatic field must satisfy $\nabla \times \mathbf{E} = \mathbf{0}$, which implies the existence of a potential function $-V$ that satisfies

$$\mathbf{E} = \nabla(-V) = -\nabla V.$$

The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). To solve for V, integrate both sides along a path between two points in space with position vectors, $\mathbf{a}$ and $\mathbf{b}$, and use the fundamental theorem for gradients.

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}_0 &= -\int_{\mathbf{a}}^{\mathbf{b}} \nabla V \cdot d\mathbf{l}_0 \\ &= -[V(\mathbf{b}) - V(\mathbf{a})] \\ &= V(\mathbf{a}) - V(\mathbf{b})\end{aligned}$$

In this context $\mathbf{a}$ is the position vector for the reference point (taken to be infinity $\boldsymbol{\infty}$ here), and $\mathbf{b}$ is the position vector $\mathbf{r}$ for the point we're interested in knowing the electric potential.

$$\int_{\boldsymbol{\infty}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}_0 = V(\boldsymbol{\infty}) - V(\mathbf{r})$$

The potential at the reference point is taken to be zero: $V(\boldsymbol{\infty}) = 0$.

$$\int_{\boldsymbol{\infty}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}_0 = -V(\mathbf{r})$$

Therefore, the potential at $\mathbf{r} = \langle x, y, z \rangle$ is

$$V(\mathbf{r}) = \int_{\mathbf{r}}^{\boldsymbol{\infty}} \mathbf{E} \cdot d\mathbf{l}_0.$$

According to Problem 2.12, the electric field around a uniformly charged solid ball is

$$\mathbf{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0} \dfrac{q}{R^3} r \hat{\mathbf{r}} & \text{if } r < R \\[2ex] \dfrac{1}{4\pi\epsilon_0} \dfrac{q}{r^2} \hat{\mathbf{r}} & \text{if } r > R \end{cases}.$$

Since the electric field is spherically symmetric, the path taken from $\mathbf{r}$ to $\boldsymbol{\infty}$ is a radial one and parameterized by r_0, where $r \leq r_0 < \infty$.

$$V(r) = \int_r^{\infty} \mathbf{E}(r_0) \cdot d\mathbf{r}_0$$

Evaluate the dot product, substitute the formula for the electric field, evaluate the integrals, and simplify the result.

$$
\begin{aligned}
V(r) &= \int_r^\infty [E(r_0)\hat{\mathbf{r}}_0] \cdot (\hat{\mathbf{r}}_0\,dr_0) \\
&= \int_r^\infty E(r_0)\,dr_0 \\
&= \begin{cases} \displaystyle\int_r^R \frac{1}{4\pi\epsilon_0}\frac{q}{R^3} r_0\,dr_0 + \int_R^\infty \frac{1}{4\pi\epsilon_0}\frac{q}{r_0^2}\,dr_0 & \text{if } r < R \\ \displaystyle\int_r^\infty \frac{1}{4\pi\epsilon_0}\frac{q}{r_0^2}\,dr_0 & \text{if } r > R \end{cases} \\
&= \begin{cases} \displaystyle\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\left(\int_r^R r_0\,dr_0\right) + \frac{q}{4\pi\epsilon_0}\left(\int_R^\infty \frac{dr_0}{r_0^2}\right) & \text{if } r < R \\ \displaystyle\frac{q}{4\pi\epsilon_0}\left(\int_r^\infty \frac{dr_0}{r_0^2}\right) & \text{if } r > R \end{cases} \\
&= \begin{cases} \displaystyle\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\left(\frac{r_0^2}{2}\right)\bigg|_r^R + \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r_0}\right)\bigg|_R^\infty & \text{if } r < R \\ \displaystyle\frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r_0}\right)\bigg|_r^\infty & \text{if } r > R \end{cases} \\
&= \begin{cases} \displaystyle\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\left(\frac{R^2}{2} - \frac{r^2}{2}\right) + \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R}\right) & \text{if } r < R \\ \displaystyle\frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}\right) & \text{if } r > R \end{cases} \\
&= \begin{cases} \displaystyle\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3 - \left(\frac{r}{R}\right)^2\right] & \text{if } r < R \\ \displaystyle\frac{q}{4\pi\epsilon_0 r} & \text{if } r > R \end{cases}
\end{aligned}
$$

In spherical coordinates

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\overbrace{\frac{\partial V}{\partial \theta}}^{=0}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\overbrace{\frac{\partial V}{\partial \phi}}^{=0}\hat{\boldsymbol{\phi}},$$

so

$$\nabla V = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{2R}\dfrac{d}{dr}\left[3-\left(\dfrac{r}{R}\right)^2\right]\hat{\mathbf{r}} & \text{if } r<R \\ \dfrac{d}{dr}\left(\dfrac{q}{4\pi\epsilon_0 r}\right)\hat{\mathbf{r}} & \text{if } r>R \end{cases}$$

$$= \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{2R}\left[-\left(\dfrac{2r}{R^2}\right)\right]\hat{\mathbf{r}} & \text{if } r<R \\ \left(-\dfrac{q}{4\pi\epsilon_0 r^2}\right)\hat{\mathbf{r}} & \text{if } r>R \end{cases}$$

$$= \begin{cases} -\dfrac{1}{4\pi\epsilon_0}\dfrac{q}{R^3}r\hat{\mathbf{r}} & \text{if } r<R \\ -\dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} & \text{if } r>R \end{cases}$$

$$= -\mathbf{E}$$

as expected. Below is a plot of $V(r)$ versus r/R.